GEOMETRY AND ALGEBRA OF REAL FORMS OF COMPLEX CURVES

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Introduction

Let X be a non-singular, non-reducible real algebraic curve of genus $g = g(X)$. It is called orientable if its real points $\mathbb{R}(X)$ divide its complex points $\mathbb{C}(X) \supset \mathbb{R}(X)$ into 2 connected components. Consider now a set of orientable non-singular, non-reducible real algebraic curves $X_1, \ldots, X_n$ $(n > 3)$ of genus $g > 1$ such that for all $i \neq j$ X_i is non-isomorphic to X_j over $\mathbb{R}$ but it is isomorphic to X_j over $\mathbb{C}$. According to [3], $(n-4)2^{n-3} \leqslant g-1$ and there exists $\{X_1, \ldots, X_n\}$ such that $(n-4)2^{n-3} = g-1$. According to Harnak theorem, $\mathbb{R}(X_i)$ form $|X_i| \leqslant g+1$ simple closed contours (ovals).

In this paper we prove that

$$\sum_{i=1}^{n} |X_i| \leqslant 2g - (n-9)2^{n-3} - 2 \leqslant 2g + 30$$

and these estimates are exact.

For $n = 3$ and 4 it was proved in [4].

The proof is based on a detal description of real forms of complex algebraic curves. This description has self-dependent importance . By our conditions all complex algebraic curves $P_i = \mathbb{C}(X_i)$ are isomorphic to a complex algebraic curve P of genus g. Consider biholomorphic maps $\varphi_i : P_i \to P$. The involutions of complex conjugations $\tau_i' : P_i \to P_i$ give antiholomorphic involutions $\tau_i = \varphi\tau_i'\varphi^{-1} : P \to P$. They generate a finite group $W : P \to P$. In §1 following [5] we prove that W is a Coxeter group. In §2 we give a complete description of all such pairs (P, W). In §3, using the results of §1, §2, and the classification of finite Coxeter groups we prove that

$$\sum_{i=1}^{n} |X_i| \leqslant 2g - (n-9)2^{n-3} - 2.$$

Farther, using an example of Singerman [8], we construct families of orientable curves $\{X_i, \ldots, X_n\}$ such that

$$\sum_{i=1}^{n} |X_i| = 2g - (n-9)2^{n-3} - 2.$$

Some of these results were announced in [6, 7].

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1. Real equipments and Coxeter groups

Let P be a complex algebraic curve, that is a compact Riemann surface of genus $g(P)$. An antiholomorphic involution $\tau : P \to P$ is called *a real form* of P [4]. It gives *a real algebraic curve* (P, τ) [1] with *real points*

$$P^\tau = \{p \in P | \tau p = p\}.$$

It is obvious that if the set $P^{\tau_1} \cap P^{\tau_2}$ is infinite then $\tau_1 = \tau_2$.

A real form τ is called *orientable* if $P/ < \tau >$ is an orientable surface. In this case P^τ divides P into 2 connected components.

A finite group W generated by real (orientable) forms is called *a real (orientable) equipment* of P.

Let W be a real orientable equipment of P. Denote by $[W]$ the set of all real orientable forms $\tau \in W$. The closure C of a connected component of $P \setminus \cup_{\tau \in [W]} P^\tau$ is called a *camera* of W. Let us consider some camera C. The *basis* of C is the set S_C of all $\sigma \in [W]$ such that the set $P^\sigma \cap C$ is infinite.

Lemma 1.1. *The basis S_C generates W. Each $\tau \in [W]$ is conjugated to some $\sigma \in S_C$.*

*Proof:*Let $\tau \in [W]$. Let us consider the group W_C generated by S_C. The set

$$\widetilde{P} = \bigcup_{\widetilde{w} \in W_C} \widetilde{w} C \subset P$$

is compact. It has no boundary and therefore $\widetilde{P} = P$. Thus there exists $\widetilde{w} \in W_C$ such that the set $\widetilde{w}C \cap P^\tau$ is infinite. Furthermore

$$\widetilde{w}C \cap P^\tau \subset \partial(\widetilde{w}C) = \widetilde{w}(\partial C) \subset \widetilde{w}(\bigcup_{\sigma \in S_C} P^\sigma).$$

Therefore there exists $\sigma \in S_C$ such that the set $P^\tau \cap \widetilde{w}(P^\sigma)$ is infinite. Thus the set

$$P^\tau \cap P^{\widetilde{w}\sigma\widetilde{w}^{-1}}$$

is infinite and $\tau = \widetilde{w}\sigma\widetilde{w}^{-1}$. It follows that W_C contains $[W]$ and therefore S_C generates W. $\square$

Let $\sigma \in [W]$ and W_σ be the set of all $w \in W$ such that C and $w(C)$ belong to the same connected component of $P \setminus P^\sigma$.

Lemma 1.2. *Let $\sigma_1, \sigma_2 \in S_C$, $w \in W_{\sigma_1}$ and $w\sigma_2 \notin W_{\sigma_1}$. Then $w\sigma_2 = \sigma_1 w$.*

Proof: The sets $w\sigma_2(C)$ and C belong to opposite connected components of $P \setminus P^{\sigma_1}$. The sets C and $w(C)$ belong to the same connected component of $P \setminus P^{\sigma_1}$. Thus $w\sigma_2(C)$ and $w(C)$ belong to the opposite connected components of $P \setminus P^{\sigma_1}$. Therefore $\sigma_2(C)$ and C belong to the opposite connected components of $P \setminus P^{w^{-1}\sigma_1 w}$ and hence

$$\sigma_2(C) \cap C \subset P^{w^{-1}\sigma_1 w}.$$

On the other hand

$$\sigma_2(C) \cap P^{\sigma_2} = C \cap P^{\sigma_2}$$

and thus the set

$$\sigma_2(C) \cap C = \sigma_2(C) \cap C \cap P^{\sigma_2} = C \cap P^{\sigma_2}$$

is infinite. Therefore the set

$$P^{\sigma_2} \cap P^{w^{-1}\sigma_1 w} \supset \sigma_2(C) \cap C$$

is infinite and $\sigma_2 = w^{-1}\sigma_1 w$. $\square$

Let $l(w)$ be the least l such that $w = \sigma_1 \dots \sigma_l$, where $\sigma_i \in S_C$.

Theorem 1.1 [5]. *Let W be a real orientable equipment of P and C its camera. Then: 1)Pair (W, S_C) is a Coxeter system that is to say $S_C = \{\sigma_1, \dots, \sigma_n\}$ generates W with defining relations $\sigma_i^2 = 1$, $(\sigma_i\sigma_j)^{m_{ij}} = 1$ for some integers m_{ij}; 2)C is a fundamental region of W.*

Proof: It is obvious that $1 \in W_\sigma$ and $W_\sigma \cap \sigma W_\sigma = \emptyset$ for $\sigma \in S_C$. It follows from $[2, IY, \S 1, n^o 7]$ that these relations and the proposition of lemma 1.2 give that (W, S_C) is a Coxeter system and

$$W_\sigma = \{w \in W | l(\sigma w) > l(w)\}.$$

Thus if $w \in W$ and $wC = C$ then

$$w \in \bigcap_{\sigma \in S_C} W_\sigma = \{w \in W | l(\sigma w) > l(w) \quad \text{for all} \quad \sigma \in S_C\} = 1.$$

Since $C \cap w(\partial C) \subset \partial C$, we see that C is a fundamental region of W.$\square$

Corollary 1.1. *Let W be a real orientable equipment of P and C its camera. Then: 1) All fixed points of all $w \in W$ belong to $\bigcup_{\tau \in [W]} P^\tau$; 2) If a real form $\tau \in W$ has real points then it is conjugated to some $\sigma \in S_C$; 3) If order of $w \in W$ is more that 5 and w has a fixed point, then w generates a normal subgroup of W.*

Proof: 1) Suppose $w \in W$ has a fixed point $p \in P \setminus \bigcup_{\tau \in [W]} P^\tau$. Since C is a fundamental region of W and

$$W(\bigcup_{\tau \in [W]} P^\tau) = \bigcup_{\tau \in [W]} P^\tau,$$

there exists $h \in W$ such that hwh^{-1} has a fixed point in $C \setminus \partial C$. Hence $hwh^{-1}(C) = C$. Therefore $hwh^{-1} = 1$ and $w = 1$. 2) Let $\tau \in W$ be a real form with real points. Then, by 1), $\tau \in [W]$. It follows from this and lemma 1.1 that τ is conjugated to some $\sigma \in S_C$. 3) Let $wp = p$. Consider a camera $C \ni p$. Let $S_C = \{\sigma_1, \dots, \sigma_n\}$. It follows from 1) that $p \in P^{\sigma_i} \cap P^{\sigma_j}$. Thus $w = (\sigma_i\sigma_j)^k$ and order of $\sigma_i\sigma_j$ is greater that 5. It follows from this and the classification of Coxeter systems [2, VI, §4] that $\sigma_i\sigma_j$ generates a normal subgroup of W. Therefore w generates a normal subgroup of W.

2. Topological classification and uniformization of equipments

Let (W, S) be a Coxeter system and $m_1, \dots, m_k$ be positive integer numbers. We shall say that a set (W, S, T) is $(m_1, \dots, m_k)$ - *swelling Coxeter system* if

$$T = \{\sigma(i,j) \in S | i = 1, \dots, k, j \in Z\},$$

where

$$\bigcup_{ij} \sigma(i,j) = S, \quad \sigma(i, j + m_i) = \sigma(i,j), \text{ and } \quad \sigma(i,j) \neq \sigma(i, j+1) \text{ if } m_i > 1.$$

We say that a $(m_1^1, \dots, m_k^1)$ - swelling Coxeter system (W^1, S^1, T^1) is *isomorphic* to a $(m_1^2, \dots, m_k^2)$ - swelling Coxeter system (W^2, S^2, T^2) if there exists a permutation $\eta : \{1, \dots, k\} \to \{1, \dots, k\}$, integers $t_1, \dots, t_k \in Z$, and an isomorphism $\psi : W^1 \to W^2$ such that $m_i^1 = m_{\eta(i)}^2$ and

$$\psi(\sigma^1(i,j)) = \sigma^2(\eta(i), j + t_i),$$

where $T^l = \{\sigma^l(i,j)\}$.

Let us now associate with every real orientable equipment (P, W) some swelling Coxeter system.

Let $C \subset P$ be a camera of W and $a_1, \dots, a_k$ be connected components of ∂C. The complex structure on P gives an orientation on C. This orientation on C gives the orientations on a_i.

If a_i is not oval thus points of intersections of ovals divide a_i on segments $l_i^1, \dots, l_i^{m_i}$. We label the segments by the index j in order that $l_i^j \cap l_i^{j+1} \neq \emptyset$, and the ordering of the segments $l_i^1, l_i^2, \dots, l_i^{m_i}$ gives the contour a_i with the given orientation on it.

If a_i is an oval that put us $m_i = 1$, $l_i^1 = a_i$. For any l_i^j it exists $\sigma(i,j) \in S_C$ such that

$$l_i^j \subset P^{\sigma(i,j)}.$$

Moreover $\sigma(i,j) \neq \sigma(i, j+1)$ and $\sigma(i,1) \neq \sigma(i, m_i)$ for $m_i \neq 1$. Put

$$T_C = \{\sigma(i,j) | i = 1, \dots, k, \quad j \in Z\},$$

where $\sigma(i, j + nm_i) = \sigma(i,j)$ for $n \in Z$. It is obvious that (W, S_C, T_C) is a $(m_1, \dots, m_k)$ - swelling Coxeter system.

Two real orientable equipments (P^1, W^1) and (P^2, W^2) are called *topological equivalent* if there exists a homeomorphism $\varphi : P^2 \to P^1$ such that $W^2 = \varphi W^1 \varphi^{-1}$.

Theorem 2.1. *Real orientable equipments (P^1, W^1) and (P^2, W^2) are topologically equivalent if and only if*

$$g(P^1/W^1) = g(P^2/W^2)$$

and there exist cameras $C^l \subset P^l$ of W^l such that the swelling Coxeter systems

$$(W^1, S_{C^1}, T_{C^1}); \quad (W^2, S_{C^2}, T_{C^2})$$

are isomorphic.

Proof: Let (P^1, W^1) and (P^2, W^2) be topologically equivalent and $\varphi : P^1 \to P^2$ be a homeomorphism such that $W^2 = \varphi W^1 \varphi^{-1}$. Let $C^1 \subset P^1$ be a camera of W^1 and $C^2 = \varphi(C^1)$. Consider a homomorphism $\psi : W^1 \to W^2$ such that $\psi(w) = \varphi w \varphi^{-1}$. Then it is obvious, that $g(P^1/W^1) = g(P^2/W^2)$ and ψ gives an isomorphism between (W^1, S_{C^1}, T_{C^1}) and (W^2, S_{C^2}, T_{C^2}).

Let us now suppose $g(P^1/W^1) = g(P^2/W^2)$ and (W^1, S_{C^1}, T_{C^1}), (W^2, S_{C^2}, T_{C^2}) be isomorphic swelling Coxeter systems for some cameras $C^l \subset P^l$. The boundaries of C^l consist of segments

$$l_{ij}^l \subset P^{\sigma^l(i,j)},$$

where $T_{C^l} = \{\sigma^l(i,j)\}$. The isomorphism

$$\psi : (W^1, S_{C^1}, T_{C^1}) \to (W^2, S_{C^2}, T_{C^2})$$

gives a correspondence $(i, j) \mapsto (\eta(i), \xi(j))$ such that

$$\psi(\sigma^1(i,j)) = \sigma^2(\eta(i), \xi(j))$$

and

$$\sigma^2(\eta(i), \xi(j+1)) = \sigma^2(\eta(i), \xi(j) + 1).$$

Thus there exists a homeomorphism $\widetilde{\varphi} : C^1 \to C^2$ such that

$$\widetilde{\varphi}(l_{ij}^1) = l_{\eta(i)\xi(j)}^2.$$

Consider now the homeomorphism $\varphi : P^1 \to P^2$, where

$$\varphi(p) = \psi(w)\widetilde{\varphi}(w^{-1}p)$$

for $p \in wC^1$. Then $W^2 = \varphi W^1 \varphi^{-1}$. $\square$

Let (W, S, T) be a $(m_1, \ldots, m_k)$ - swelling Coxeter system and $T = \{(\sigma(i,j)\}$. Let $n(i,j)$ be order of $(\sigma(i,j) \cdot \sigma(i,j+1))$. Put

$$\mu_g = \mu_g(W, S, T) = 4g + 2k - 4 + \sum_{i=1}^{k} \sum_{j=1}^{m_i} (1 - \frac{1}{n(i,j)}).$$

Denote

$$\Lambda = \Lambda((W, S, T), g)$$

the Riemann sphere if $\mu_g = 0$, the complex plane $\mathbb{C}$ if $\mu_g = 1$, and the upper half-plane

$$\{z \in \mathbb{C} | \text{Im } z > 0\}$$

if $\mu > 1$. Let $\overline{\text{Aut}}(\Lambda)$ be the group of all holomorphic and all antiholomorphic automorphisms of Λ and

$$\text{Aut}(\Lambda) \subset \overline{\text{Aut}}(\Lambda)$$

be the subgroup of holomorphic automorphisms.

Consider a discrete group $G \subset \overline{\text{Aut}}(\Lambda)$ and an epimorphism $\psi : G \to W$. The pair (G, ψ) is called a *g-planar realization of* (W, S, T) if G has generators

$$\{a_\alpha, b_\alpha \in \text{Aut}(\Lambda) \quad (\alpha = 1, \ldots, g), \quad c_i \in \text{Aut}(\Lambda) \quad (i = 1, \ldots, k),$$

$$\sigma_{ij} \notin \text{Aut}(\Lambda)(i = 1, \ldots, k, j = 1, \ldots, m_i + 1)\},$$

generating G with defining relations

$$\prod_{\alpha=1}^{g} [a_\alpha b_\alpha] \prod_{i=1}^{k} c_i = 1, \quad \sigma_{ij}^2 = 1, \quad (\sigma_{ij} \cdot \sigma_{ij+1})^{n(i,j)} = 1, \quad \sigma_{i1} c_i \sigma_{im_i+1} = c_i$$

and moreover

$$\psi(\sigma_{ij}) = \sigma(i,j), \quad \psi(a_\alpha) = \psi(b_\alpha) = \psi(c_i) = 1.$$

Here $[ab] = aba^{-1}b^{-1}$ is the commutator of the elements $a, b \in G$.

Lemma 2.1. *Let (G, ψ) be a g-planar realization of (W, S, T), $P = \Lambda/ker\,\psi$. $W_P = G/ker\,\psi$. Then (P, W_P) is a real orientable equipment and there exists a camera $C \subset P$ such that the swelling Coxeter system (W_P, S_C, T_C) is isomorphic to (W, S, T).*

Proof: Put $\widetilde{P} = \Lambda/G$. Let $F : G \to W_P$, $\Phi : \Lambda \to P$, $\varphi : P \to \widetilde{P}$ and $\tilde{\Phi} = \varphi\Phi : \Lambda \to \widetilde{P}$ be the natural projections. It follows from [9, Ch4] that all critical values of $\widetilde{\Phi}$ belong to $\partial\widetilde{P}$. Moreover the fundamental group of $\widetilde{P}$ is generated by the images of a_α, b_α, c_i. Thus φ is a homeomorphism on each connected component of $\varphi^{-1}(\widetilde{P} \setminus \partial\widetilde{P})$ and the closing of each one is a fundamental region of W_P. It follows from [9, Ch4] that G has a fundamental region B such that

$$\partial\widetilde{\Phi}(B) = \widetilde{\Phi}(\bigcup_{ij}\{z \in B | \sigma_{ij}z = z\}).$$

Thus the $C = \Phi(B)$ is a fundamental region of W_P and

$$\partial C \subset \bigcup_{ij} P^{F(\sigma_{ij})}.$$

Put

$$S_C^1 = \{\sigma_1, \ldots, \sigma_n\} = F(\{\sigma_{ij}\ (i = 1, \ldots, k;\ j = 1, \ldots, m_i\}).$$

Then (W_P, S_C^1) is a Coxeter system and

$$\partial C \subset \bigcup_{i=1}^{m} P^{\sigma_i}.$$

Let us prove that each real form σ_s is orientable. Consider

$$W_s = \{w \in W_P | l(\sigma_s w) > l(w)\},$$

where $l(w)$ is the least l such that $w = \sigma_{i_1} \cdots \sigma_{i_l}$. Put

$$P_1 = W_s C \quad \text{and} \quad P_2 = (W \setminus W_s)C.$$

Let

$$p \in P_1 \cap P_2 = \partial P_1 = \partial P_2.$$

Then $p \in w_1 C \cap w_2 C$, where $w_1 \in W_s$, $w_2 \in W \setminus W_s$. Put $p_0 = w_1^{-1} p \in \partial C$. Then $\sigma_t p_0 = p_0$ for some $\sigma_t \in S_C^1$. Thus

$$(w_1 \sigma_t w_1^{-1}) p = p \quad \text{and} \quad (w_1 \sigma_t w_1^{-1})(w_1 C) = w_2 C.$$

Therefore $w_1 \sigma_t = w_2 \notin W_s$ and according to [2, IY, §1 $n^0 7$] $w_1 \sigma_t w_1^{-1} = \sigma_s$. Thus

$$p \in P^{\sigma_s} \quad \text{and} \quad P_1 \cap P_2 \subset P^{\sigma_s}.$$

Therefore P^{σ_s} divides P into 2 connected components.

Thus (P, W_P) is a real orientable equipment with the camera C, $S_C^1 = S_C$ and ψ gives an isomorphism between (W_P, S_C, T_C) and (W, S, T).□

Theorem 2.2. *For each $(m_1, \ldots, m_k)$-swelling Coxeter group (W, S, T) and $g \geq 0$ there exists a g - planar realization of (W, S, T) and a real orientable equipment (P, W_P) with a camera $C \subset P$ such that $g(P/W_P) = g$ and the swelling Coxeter system (W_P, S_C, T_C) is isomorphic to (W, S, T).*

Proof: It follows from [9, Ch 4] that there exists a discrete group

$$G \in \overline{\mathrm{Aut}}(\Lambda) \quad (\Lambda = \Lambda((W, S, T), g)$$

with generators

$$\{a_\alpha, b_\alpha \in \mathrm{Aut}(\Lambda) \quad (\alpha = 1, \ldots, g), \qquad c_i \in \mathrm{Aut}(\Lambda) \quad (i = 1, \ldots, k),$$

$$\sigma_{ij} \notin \mathrm{Aut}(\Lambda) \quad (i = 1, \ldots, k, \quad j = 1, \ldots, m_i + 1)\}$$

and the defining relations

$$\sigma_{ij}^2 = 1, \quad (\sigma_{ij} \cdot \sigma_{ij+1})^{n(i,j)} = 1,$$

$$\prod_{\alpha=1}^{g} [a_\alpha b_\alpha] \prod_{i=1}^{k} c_i = 1, \quad \sigma_{i1} c_i \sigma_{im_i+1} = c_i,$$

where $n(i, j)$ is the order of

$$(\sigma(i, j) \cdot \sigma(i, j + 1))$$

and

$$T = \{\sigma(i,j)\}.$$

Put now

$$\psi(a_\alpha) = \psi(b_\alpha) = \psi(c_i) = 1$$

and

$$\psi(\sigma_{ij}) = \sigma(i,j).$$

Then (G, ψ) is a g- planar realization of (W, S, T) and theorem 2.2 follows from lemma 2.1. $\square$

We say that real equipments (P_1, W_1) and (P_2, W_2) are *isomorphic* if there exists a holomorphic map $\varphi : P_1 \to P_2$ such that $W_2 = \varphi W_1 \varphi^{-1}$.

Theorem 2.3. *Each real orientable equipment is isomorphic to*

$$(\Lambda/\ker\psi, G/\ker\psi),$$

where (G, ψ) is a g-planar realization of some swelling Coxeter system.

Proof: Let (P, W_P) be a real orientable equipment, $g = g(P/W_P)$ and C be some camera of W_P. It follows from theorem 2.2 that there exist a g- planar realization (G_0, ψ_0) of the swelling Coxeter system (W_P, S_C, T_C). It follows from lemma 2.1 and theorem 2.1 that there exists a homeomorphism

$$\varphi : P \to \Lambda/\ker\psi_0$$

such that

$$\varphi W_P \varphi^{-1} = G_0/\ker\psi_0.$$

Consider a uniformization $\Phi : \Lambda \to P$ and a natural projection

$$\Phi_0 : \Lambda \to \Lambda/\ker\psi_0.$$

According to [9, Ch.5], there exists a homeomorphism $\widetilde{\varphi} : \Lambda \to \Lambda$ such that $\Phi_0\widetilde{\varphi} = \varphi\Phi$. Put $G = \widetilde{\varphi}^{-1} G_0 \widetilde{\varphi}$ and $\psi = \psi_0 F$, where $F : G \to G_0$ and $F(w) = \widetilde{\varphi} w \widetilde{\varphi}^{-1}$. Then (G, ψ) is a g-realization of (W_P, S_C, T_S) and Φ gives a isomorphism between $(\Lambda/\ker\psi,\ G/\ker\psi)$ and (P, W_P). $\square$

3. Total number of ovals

We shall use

Lemma 3.1.[4] *Let $W : P \to P$ be a finite group of autohomeomorphisms of a compact orientable surface P. Then there exists a complex structure on P such that W consists of holomorphic and antiholomorphic homeomorphisms.*

It follows from Harnak's theorem that the set P^α of the fixed points of a real form $\alpha : P \to P$ consists of $|\alpha| \leqslant g(P) + 1$ simple closed contours (ovals). For a real orientable equipment (P, W) we put

$$h(P, W) = \sum_{\alpha \in [W]} |\alpha| - 2g(P)$$

and

$$P^W = \bigcup_{\alpha \in [W]} P^\alpha.$$

A real orientable equipment (P, W) is called *commutative* if W is a commutative group. Put $h(n, g) = \max\{h(P, W)\}$, where max is taken by all commutative equipments (P, W) such that $g(P) = g$ and $[W]$ consists of n elements.

Consider a function $f(2) = 2$, $f(n) = -(n-9)2^{n-3} - 2$ for $n > 2$. Our next goal is the proof that $h(n, g) \leqslant f(n)$ for $n > 1$.

Lemma 3.2. *Suppose $h(n', g') \leqslant f(n')$ for all $g' < g$ and $n' > 2$. Let (P, W) be a commutative equipment such that $g(P) = g$, $[W] = (\alpha_1, \dots, \alpha_n)$, where $n > 2$, and some connected component of P^W belongs to P^{α_n}. Then $h(P, W) < f(n)$.*

Proof: Let a be a connected component of P^W and $a \subset P^{\alpha_n}$. Then a is an oval of α_n and $A = \bigcup_{w \in W} w(a)$ consists of 2^{n-1} non-intersecting contours. If $P \setminus A$ is connected, we squeeze to a point each boundary contour of $P \setminus A$. Thus we obtain a compact surface P', where $g' = g(P') = g - 2^{n-1}$. The forms $\alpha_1, \dots, \alpha_n$ give involutions $\alpha'_i : P' \to P'$. It follows from lemma 3.1 that it exists a complex structure on P' such that $\alpha'_1, \dots, \alpha'_n$ are real forms of P'. They generate a commutative equipment such that

$$\sum_{i=1}^{n} |\alpha'_i| = \sum_{i=1}^{n} |\alpha_i| - 2^{n-1}.$$

Therefore

$$h(P,W)=\sum_{i=1}^{n}|\alpha_i|-2g=\sum_{i=1}^{n}|\alpha_i'|+2^{n-1}-2g'-2^n=(\sum_{i=1}^{n}|\alpha_i'|-2g')-2^{n-1}<h(n,g')\leqslant f(n).$$

Let now $P\setminus A$ be disconnected. Then $P\setminus A$ consists of two connected components P_1 and P_2. Contract to a point each boundary contour of P_1 to produce a compact surface P' of genus

$$g'=g(P')=\frac{1}{2}(g-2^{n-1}+1).$$

The forms $\alpha_1,\dots,\alpha_{n-1}$ give involutions $\alpha_i':P'\to P'$. Consider a complex structure on P' such that $\alpha_1',\dots,\alpha_{n-1}'$ are real forms of P'. They generate a commutative equipment such that

$$\sum_{i=1}^{n}|\alpha_i'|=\frac{1}{2}(\sum_{i=1}^{n}|\alpha_i|-2^{n-1}).$$

Therefore

$$h(P,W)=\sum_{i=1}^{n}|\alpha_i|-2g=2\sum_{i=1}^{n-1}|\alpha_i'|+2^{n-1}-4g'-2^n+2=$$

$$2(\sum_{i=1}^{n-1}|\alpha_i'|-2g')-2^{n-1}+2\leqslant 2f(n-1)-2^{n-1}+2.$$

Thus if $n=3$, then

$$h(P,W)\leqslant 4-4+2<4=f(3).$$

If $n>3$, then

$$h(P,W)\leqslant 2(-(n-10)2^{n-4}-2)-2^{n-1}+2=-2^{n-3}(n-6)-2<f(n).\square$$

Lemma 3.3. *Suppose $h(n',g')\leqslant f(n')$ for all $g'<g$ and $n'>2$. Let (P,W) be a commutative equipment such that $g(P)=g$, $[W]=(\alpha_1,\dots,\alpha_n)$, where $n>2$, and some connected component of P^W belongs to $P^{\alpha_{n-1}}\cup P^{\alpha_n}$. Then $f(P,W)<f(n)$.*

Proof: Let a be a connected component of P^W and $a\subset P^{\alpha_{n-1}}\cup P^{\alpha_n}$. If $a\subset P^{\alpha_{n-1}}$ or $a\subset P^{\alpha_n}$, then lemma 3.3 follows from lemma 3.2. Let $a\not\subset P^{\alpha_{n-1}}$ and $a\not\subset P^{\alpha_n}$. In this case a consists of some number m of ovals from $P^{\alpha_{n-1}}$ and the same number m of ovals from P^{α_n}.

Moreover each of these ovals contains exactly 2 points of $P^{\alpha_{n-1}} \cap P^{\alpha_n}$. Thus $A = \cup_{w \in [W]} w(a)$ consists of $m \cdot 2^{n-2}$ ovals of α_{n-1} and $m \cdot 2^{n-2}$ ovals of α_n.

Suppose $P \setminus A$ is connected. Then we contract to a point each boundary component of $P \setminus A$. Thus we obtain a compact surface P' of genus $g' = g(P') = g - (m+2) \cdot 2^{n-2}$. The forms $\alpha_1, \dots, \alpha_n$ give involutions $\alpha'_i : P' \to P'$. Consider a complex structure on P' such that $\alpha'_1, \dots, \alpha'_n$ are real forms of P'. They generate a commutative equipment such that

$$\sum_{i=1}^{n} |\alpha'_i| = \sum_{i=1}^{n} |\alpha_i| - m \cdot 2^{n-1}.$$

Therefore

$$h(P, W) = \sum_{i=1}^{n} |\alpha_i| - 2g = \sum_{i=1}^{n} |\alpha'_i| + m \cdot 2^{n-1} - 2g' - m \cdot 2^{n-1} - 2^{n+1} < h(n, g') \leqslant f(n).$$

If $P \setminus A$ is disconnected then it forms 4 connected components. Let P_1 be one of them. Contract to a point each boundary component of P_1 to obtain a compact surface P' with

$$g' = g(P') = \frac{1}{4}(g - (m-1)2^{n-2} - 4(2^{n-2} - 1)).$$

The forms $\alpha_1, \dots, \alpha_{n-2}$ give involutions $\alpha'_i : P' \to P'$. Consider a complex structure on P' such that $\alpha'_1, \dots, \alpha'_{n-2}$ are real forms of P'. They generate a commutative equipment such that

$$\sum_{i=1}^{n-2} |\alpha'_i| = \frac{1}{4}(\sum_{i=1}^{n} |\alpha_i| - m \cdot 2^{n-1}).$$

Thus

$$h(P, W) = \sum_{i=1}^{n} |\alpha_i| - 2g = 4\sum_{i=1}^{n-2} |\alpha'_i| + m \cdot 2^{n-1} - 8g' - (m-1)2^{n-1} - 8(2^{n-2} - 1) =$$

$$4(\sum_{i=1}^{n-2} |\alpha'_i| - 2g') - 3 \cdot 2^{n-1} + 8 = 4f(n-2) - 3 \cdot 2^{n-1} + 8.$$

If $n > 4$ then

$$h(P, W) \leqslant 4(-(n-11)2^{n-5} - 2) - 3 \cdot 2^{n-1} + 8$$

$$= 2^{n-3}(-n + 11 - 12) - 8 + 8 = 2^{n-3}(-n-1) < -2^{n-3}(n-9) - 2 = f(n).$$

If $n=4$ then

$$h(P,W) \leqslant 4\cdot 2 - 3\cdot 2^3 + 8 = -8 < f(4).$$

If $n=3$ then

$$P^{\alpha_1} \cap (P^{\alpha_2} \cup P^{\alpha_3}) = \emptyset$$

and lemma 3.3 follows from lemma 3.2. $\square$

Lemma 3.4. *Let (P,W) be a commutative equipment such that $[W]=(\alpha_1,\ldots,\alpha_n)$, where $n>2$. Then $(n-4)2^{n-3} \leqslant g-1$. Moreover if every connected component of P^W does not belong to $P^{\alpha_i} \cup P^{\alpha_j}$ for each i,j, then*

$$\sum_{i=1}^{n} |\alpha_i| \leqslant 2g(P) - (n-9)2^{n-3} - 2.$$

Proof: Let $C \subset P$ be a camera of W and $\widetilde{a}_1,\ldots,\widetilde{a}_k$ be its boundary contours. The contour $\widetilde{a}_i$ consists of segments

$$\ell_{i1},\ldots,\ell_{im_i},$$

where

$$\ell_{ij} \subset P^{\sigma(i,j)},\ \sigma(i,j) \in [W]$$

and

$$\sigma(i,j) \neq \sigma(i,j+1),\ \sigma(i,1) \neq \sigma(i,m_i).$$

Let t_i be the number of distinct elements between

$$\sigma(i,1),\ldots,\sigma(i,m_i).$$

Our conditions give $t_i \geqslant 3$. Moreover

$$\sum_{i=1}^{k} t_i \geqslant n.$$

Put

$$\sigma(i,j+nm_i) = \sigma(i,j)$$

for $n \in Z$. Consider

$$L_i^1 = \{\ell_{ij} | \sigma(i, j-1) = \sigma(i, j+1)\},$$

$$L_i^2 = \{\ell_{ij} | \sigma(i, j-1) \neq \sigma(i, j+1)\}.$$

Let s_i be the number of elements in L_i^2. It follows from $t_i \geqslant 3$ that $s_i \geqslant t_i - 1$.

Let $\widetilde{P} = P/W$ and $\varphi : P \to \widetilde{P}$ be the natural projection. Then $\varphi^{-1}(\ell_{ij})$ consists of ovals of $\sigma(i,j)$. The number of these ovals is 2^{n-2} if $\ell_{ij} \in L_i^1$ and 2^{n-3} if $\ell_{ij} \in L_i^2$. Thus

$$\sum_{i=1}^{n} |\alpha_i| = \sum_{i=1}^{k} (s_i \cdot 2^{n-3} + (m_i - s_i) \cdot 2^{n-2}) = \sum_{i=1}^{k} (m_i \cdot 2^{n-2} - s_i \cdot 2^{n-3}) \leqslant$$

$$\sum_{i=1}^{k} (m_i \cdot 2^{n-2} - (t_i - 1) 2^{n-3}) = (\sum_{i=1}^{k} m_i) \cdot 2^{n-2} + k \cdot 2^{n-3} - (\sum_{i=1}^{k} t_i) \cdot 2^{n-3} \leqslant$$

$$(\sum_{i=1}^{k} m_i) \cdot 2^{n-2} + (k - n) \cdot 2^{n-3}.$$

On the other hand, it follows from theorem 2.3 that

$$(P, W) = (\Lambda/\text{Ker } \psi, \ G/\text{Ker } \psi),$$

where (G, φ) is a $g-$planar realization the swelling Coxeter system $(W, [W], T)$, and

$$T = \{\sigma(i,j), i = 1, \ldots, k, j \in Z\}.$$

It follows from Riemann-Hurwitz's theorem ([9], 4.14.21) that

$$4g - 4 = 2^n (4\widetilde{g} - 4 + 2k + \frac{1}{2} \sum_{i=1}^{k} m_i),$$

where $g = g(P), \quad \widetilde{g} = g(\widetilde{P})$. Thus

$$g - 1 \geqslant 2^{n-2}(-2 + \frac{1}{2} n) = 2^{n-3}(n - 4)$$

and

$$\sum_{i=1}^{n} |\alpha_i| - 2g \leqslant (\sum_{i=1}^{k} m_i) \cdot 2^{n-2} + (k - n) \cdot 2^{n-3} - 2^{n-1}(4\widetilde{g} - 4 + 2k + \frac{1}{2} \sum_{i=1}^{k} m_i) - 2 \leqslant$$

$$\leqslant (k-n)\cdot 2^{n-3} + 2^{n+1} - k\cdot 2^n - 2 \leqslant (1-n)\cdot 2^{n-3} + 2^{n+1} - 2^n - 2 =$$

$$= -2^{n-3}(n-1-16+8) - 2 = -(n-9)\cdot 2^{n-3} - 2.\square$$

Lemma 3.5. *Let (P, W) be a real orientable equipment and $\tau_1, \ldots, \tau_n \in [W]$ (where $n > 2$) be non-conjugate. Then there exists a commutative equipment $W' \subset W$ such that*

$$[W'] = (\alpha_1, \ldots, \alpha_n), \quad \alpha_i \neq \alpha_j$$

and

$$\sum_{i=1}^{n} |\tau_i| \leqslant \sum_{i=1}^{n} |\alpha_i|.$$

Proof: Let $\widetilde{W}$ be the real orientable equipment generated by $\tau_1, \ldots, \tau_n$ and C its camera. It follows from lemma 1.1 that there exist $\beta_1, \ldots, \beta_n \in S_C$ such that $\beta_i = w_i \tau_i w_i^{-1}$, $w_i \in \widetilde{W}$. Let $\widetilde{W}'$ be the real orientable equipment, generated by $\widetilde{S}' = (\beta_1, \ldots, \beta_n)$. Put $\alpha_j = \beta_j$ if β_j belongs to the center of $\widetilde{W}'$.

Let us now assume that β_j does not belong to the center of $\widetilde{W}'$. It follows from theorem 1.1 that $(\widetilde{W}', \widetilde{S}')$ is a Coxeter system. Moreover $\beta_1, \ldots, \beta_n$ are non-conjugated in $\widetilde{W}'$. We observe, using the classification of Coxeter systems [2, VI, §4], that there exists only one $\beta_i \in \widetilde{S}'$ such that $\beta_i\beta_j \neq \beta_j\beta_i$. For similar reasons $\beta_i\beta_k = \beta_k\beta_i$ if $k \neq j$. The order $2m$ of $\beta_i\beta_j$ is even because β_i and β_j are non-conjugated in $\widetilde{W}'$. Put $\gamma = (\beta_i\beta_j)^m$. Then γ belongs to the center of $\widetilde{W}'$. For $|\beta_i| \geqslant |\beta_j|$ we put $\alpha_i = \beta_i$, $\alpha_j = \gamma\beta_i$. For $|\beta_j| \geqslant |\beta_i|$ we put $\alpha_j = \beta_j$, $\alpha_i = \gamma\beta_j$. Then $\alpha_1, \ldots, \alpha_n$ generate a commutative equipment W' and

$$\sum_{i=1}^{n} |\alpha_i| \geqslant \sum_{i=1}^{n} |\tau_i|.\square$$

Theorem 3.1. *Let (P, W) be a real orientable equipment, $g(P) = g$ and $\tau_1, \ldots, \tau_n \in [W]$ (where $n > 2$) be non-conjugated in W. Then $(n-4)2^{n-3} \leqslant g - 1$ and*

$$\sum_{i=1}^{n} |\tau_i| \leqslant 2g - (n-9)2^{n-3} - 2.$$

Proof: Due to lemma 3.5 it suffices to prove theorem 3.1 for commutative equipment. For this case it follows from lemma 3.4 that $(n-4)2^{n-3} \leqslant g-1$. We use an induction on $g = g(P)$ for to prove

$$\sum_{i=1}^{n} |\tau_i| \leqslant 2g - (n-9)2^{n-3} - 2.$$

If $g(P) = 0$ then $n = 3$ and $\sum_{i=1}^{n} |\tau_i| = 3 < 6 - 2$. Let us assume that the statement is proved for the cases $g(P) < g$. If it exists a connected component a of P^W such that $a \subset P^{\tau_i} \cup P^{\tau_j}$, then the statement of theorem 3.1 follows from lemma 3.3. Otherwise it follows from lemma 3.4. □

Theorem 3.2. *For any $n > 2$ and $m \geqslant 0$, where $n + 2m > 4$, there exists a commutative equipment (P, W) such that the forms $[W] = (\tau_1, \ldots, \tau_n)$ are non-conjugated with respect to holomorphic automorphisms of P,*

$$g(P) = 2^{n-3}(n + 2m - 4) + 1$$

and

$$\sum_{i=1}^{n} |\tau_i| = 2g(P) - (n-9)2^{n-3} - 2.$$

Proof: Consider a rectangular $(n+2m)$-gon with incommensurable sides $\ell_1, \ldots, \ell_{n+2m}$ on Lobachevskij plane Λ. Let $G \subset \overline{\mathrm{Aut}}(\Lambda)$ be the group generated by reflections σ_i in the sides ℓ_i of the polygon. Let $\widetilde{W} \cong (Z_2)^n$ be the group generated by n involutions $s_1, \ldots, s_n$. Consider the epimorphism $\psi : G \to \widetilde{W}$ such that $\psi(\sigma_i) = s_i$ for $i = 1, \ldots, n-1$, $\psi(\sigma_{n+2j}) = s_2$, $\psi(\sigma_{n+2j+1}) = s_n$ for $j = 0, ..., m-1$. Then (G, ψ) is a plan realization. It follows from lemma 2.1 that $(P, W) = (\Lambda/\mathrm{Ker}\ \psi, \quad G/\mathrm{Ker}\ \psi)$ is a commutative equipment and $[W] = (\tau, \ldots, \tau_n)$, where

$$\tau_i = \sigma_i/\mathrm{Ker}\ \psi \quad \text{for } i = 1, \ldots, n-1, \quad \tau_2 = \sigma_{n+2i}/\mathrm{Ker}\ \psi \quad \text{for } i = 0, \ldots, m,$$

$$\tau_n = \sigma_{n+2j+1}/\mathrm{Ker}\ \psi, \quad \text{for } j = 0, \ldots, m-1.$$

Riemann-Hurwitz's formula [9, 4.14.11] gives

$$g(P) = 2^{n-3}(n + 2m - 4) + 1.$$

Let $\psi : P \to P/W$ be the natural projection. Then $\psi^{-1}(\ell_i)$ forms 2^{n-3} ovals of $\tau \in [W]$ for $i = 2, \ldots, n$ and it forms 2^{n-2} ovals of $\tau \in [W]$ for $i = 1$ and $i > n$. Thus

$$\sum_{i=1}^{n} |\tau_i| = (n-1)2^{n-3} + (2m+1)2^{n-2}$$

and

$$2g(P) - \sum_{i=1}^{n} |\tau_i| = 2^{n-3}(2n + 4m - 8) + 2 - (n-1)2^{n-3} - (4m+2)2^{n-3} =$$

$$= (n-9)2^{n-3} + 2.$$

It follows from incommensurability of ℓ_i that τ_i are non-conjugated with respect to holomorphic automorphisms (i.e. isometries with respect to Lobachevskij metric) of P. $\square$

Corollary 3.1. *Let $X_1, \ldots, X_n$ $(n > 3)$ be orientable non-singular, non-reducible real algebraic curves of genus $g > 1$ such that for any $i \neq j$ X_i is non-isomorphic to X_j over $\mathbb{R}$ but isomorphic over $\mathbb{C}$. Then*

$$\sum_{i=1}^{n} |X_i| \leqslant 2g - (n-9)2^{n-3} - 2 \quad \textit{and} \quad (n-4)2^{n-3} \leqslant g - 1$$

and this estimate is attained for each n for infinite number of g.

Proof: By definition there exists a Riemann surface P and biholomorphic maps $\psi : P_i \to P$, such that $X_i = (P_i, \alpha_i)$. Then $\tau_i = \psi_i \alpha_i \psi_i^{-1}$ generate an orientable equipment of P. Thus corollary 3.1 follows from theorems 3.1 and 3.2. $\square$

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